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Original Research Article

ON DEVELOPMENT OF EXPONENTIATED FRECHET WEIBULL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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This study was a collaborative effort among all authors. Each author reviewed and approved the final version of the manuscript for publication.

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ABSTRACT
The Exponentiated Fréchet-Weibull (EF-Weibull) distribution is introduced as a flexible statistical model that extends the classical Fréchet and Weibull distributions by incorporating additional shape and scale parameters. This study derives its probability density function (PDF), cumulative distribution function (CDF), quantile function, hazard function, and moment-based properties, providing a comprehensive theoretical foundation. Maximum likelihood estimation (MLE) was employed for parameter estimation, ensuring its applicability to real-world datasets. A comparative analysis was conducted against the Exponentiated Fréchet (Exp-Fréchet), Exponentiated Weibull (Exp-Weibull), Exponentiated Exponential (Exp-Exponential), Fréchet, and Weibull distributions using log-likelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (KS) test, Anderson-Darling (AD) test, and Cramér-von Mises (CS) test. The results showed that the EF-Weibull distribution provided the best fit, achieving the highest LL and lowest AIC and BIC values, while also demonstrating superior empirical performance through KS, AD, and CS test statistics. Graphical evaluations, including histogram density plots, hazard function plots, and empirical CDF comparisons, further validated its modelling efficiency. The study concludes that the EF-Weibull distribution is an effective model for reliability analysis, survival analysis, and extreme value modelling.

Keywords
Exponentiated, Fréchet-Weibull, distribution, Applications, Distribution

INTRODUCTION
The Exponentiated Fréchet-Weibull (EF-Weibull) distribution is an advanced statistical model derived from the Exponentiated Fréchet-G (EF-G) family, as proposed by Lamya A. Baharith and Hanan H. Alamoudi (2021). This distribution provides a more flexible framework for modelling complex datasets, particularly those exhibiting extreme values and heavy-tailed behaviours. By integrating the properties of the Weibull and Fréchet distributions

within the EF-G framework, the EF-Weibull model extends the capabilities of traditional statistical distributions used in survival analysis, reliability studies, and risk assessment.

The Weibull distribution, first introduced by Wallodi Weibull (1951), has been widely employed in reliability and failure time analysis due to its ability to model various hazard rates. However, it often fails to capture extreme tail behaviour accurately. On the other hand, the Fréchet distribution, which belongs to the family of extreme value distributions, is known for its effectiveness in modelling extreme phenomena but may not always fit moderate failure rates adequately. The EF-G family, introduced to generalise probability distributions using the transformed-transformer (T-X) method, provides an efficient mechanism for introducing additional shape parameters, enhancing the flexibility of standard distributions. The EF-Weibull distribution, developed by incorporating the Weibull baseline within the EF-G framework, is thus a powerful tool for analyzing diverse real-world data. The development of generalized statistical distributions has been a significant research area in probability and statistics. Several studies have extended classical distributions to provide more adaptable models for empirical data. Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution, which introduced an additional shape parameter to increase flexibility. Gupta and Kundu (1999) further explored the exponentiated family of distributions in reliability analysis, demonstrating their superior performance in modelling lifetime data.

The Fréchet distribution, originally introduced by Maurice Fréchet in the 1920s, has been extensively used in extreme value analysis (Coles, 2001). Researchers have modified this distribution to accommodate a wider range of data structures. Nadarajah and Kotz (2003) highlighted the advantages of using extreme value distributions in finance and climate modeling. The EF-G family, introduced by Baharith and Alamoudi (2021), provided a novel approach to enhancing distributional flexibility through the T-X transformation, allowing for broader applications in statistical modeling. By incorporating the Weibull distribution into the EF-G framework, the EF-Weibull distribution further expands the applicability of these models.

Recent studies, such as Cordeiro *et al.* (2015) and Nofal *et al.* (2016), have shown that exponentiated and generalized distributions improve data fitting and inference in various fields, including reliability engineering, hydrology, and economics. These advancements highlight the relevance of the EF-Weibull model in addressing complex data challenges.

METHODOLOGY

The Exponentiated Fréchet-Weibull distribution is derived by substituting the Weibull CDF into the EF-G family framework. The CDF and PDF of the EF-G family, as defined by Baharith and Alamoudi (2021), are given as:

Cumulative Distribution Function (CDF):

$$F(x;\alpha,\theta,\gamma,\beta) = 1 - \left(1 - e^{\left(-\left(\frac{\gamma}{-log(1-G(x;\beta))}\right)^{\theta}\right)}\right)^{\alpha} \tag{1}$$

Probability Density Function (PDF):

$$f(x;\alpha,\theta,\gamma,\beta) = \alpha\theta\gamma^{\theta} g(x)\left(-log\left(1-G(x;\beta)\right)\right)^{-(\theta+1)} e^{\left(-\left(\frac{\gamma}{-log(1-G(x;\beta))}\right)^{\theta}\right)} \times \left(1 - e^{\left(-\left(\frac{\gamma}{-log(1-G(x;\beta))}\right)^{\theta}\right)}\right)^{\alpha-1} \tag{2}$$

Where $G(x;\beta)$ represents the baseline Weibull CDF.

Substituting the Weibull distribution $G(x;\beta) = 1 - e^{-(\lambda x)^{\beta}}$ into the EF-G framework, we obtain the EF-Weibull distribution's CDF and PDF as:

CDF:

$$F(x;\alpha,\theta,\gamma,\beta,\lambda) = 1 - \left(1 - e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha} \tag{3}$$

Pdf of Exponentiated Fréchet-Weibull distribution is expressed as follows:

$$f(x;\alpha,\theta,\gamma,\beta,\lambda) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\, x^{-(\beta\theta+1)} e^{-\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}}\left(1 - e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha-1} \tag{4}$$

To rewrite the Probability Density Function (PDF) of the Exponentiated Fréchet-Weibull (EF-Weibull) distribution using the Binomial expansion series:

$$(1-z)^n = \sum_{i=0}^{\infty}(-1)^i \binom{n}{i} z^i \quad (5)$$

From equation 4 and equation five we obtain

$$\left(1-e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha-1} = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha-1}{k} e^{-(k)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}$$

Given that $e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}} = z$ and $\alpha - 1 = n$

substitute $\sum_{k=0}^{\infty}(-1)^k \binom{\alpha-1}{k} e^{-(k)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}$ back to equation 4

$$f(x;\alpha,\theta,\gamma,\beta,\lambda) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\, x^{-(\beta\theta+1)} e^{-\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}} \sum_{k=0}^{\infty}(-1)^k \binom{\alpha-1}{k} e^{-(k)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}$$

Since these exponents $\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}$ and $(k)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}$ are having the same base the equation above becomes:

$$f(x;\alpha,\theta,\gamma,\beta,\lambda) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} x^{-(\beta\theta+1)} \sum_{k=0}^{\infty}(-1)^k \binom{\alpha-1}{k} e^{-(k+1)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}$$

$$let\ w_i = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha-1}{k}$$

Therefore the pdf can be expressed as:

$$f(x;\alpha,\theta,\gamma,\beta,\lambda) = \alpha\theta\gamma^{\theta}\beta\lambda^{-(\beta\theta+1)} x^{-(\beta\theta+1)} w_i e^{-(k+1)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}} \quad (6)$$

The quantile function is expressed as follows:

$$Q(q) = \frac{\gamma}{\lambda\left(-ln\left(1-(1-q)^{\frac{1}{\alpha\theta}}\right)\right)^{\beta}} \quad (7)$$

The survival function is expressed as follows:

$$S(x) = 1 - F(x)$$

$$S(x) = \left(1-e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha} \quad (8)$$

The hazard function is expressed as follows:

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\, x^{-(\beta\theta+1)} e^{-\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}} \left(1-e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{-1} \quad (9)$$

To obtain the r^{th} moment of Exponentiated Fréchet-Weibull distribution

r^{th} moment can be obtain using this formula

$$E(x^r) = \int_0^{\infty} x^r f(x;\alpha,\theta,\gamma,\beta)\,dx \quad (10)$$

Substitute equation (5) in equation (9)

$$E(x^r) = \int_0^{\infty} x^r\, \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} x^{-(\beta\theta+1)} w_i e^{-(k+1)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\,dx$$

$$E(x^r) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\int_0^{\infty} x^r\, x^{-(\beta\theta+1)} w_i e^{-(k+1)\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}} dx$$

$$E(x^r) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} w_i \int_0^{\infty} x^{r-(\beta\theta+1)}\, e^{-(k+1)}\gamma^{\theta}(\lambda x)^{-\beta\theta} dx \tag{11}$$

$$\text{let t} = (k+1)\gamma^{\theta}\beta(\lambda x)^{-\beta\theta} \tag{12}$$

$$E(x^r) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} w_i \int_0^{\infty} x^{r-(\beta\theta+1)}\, e^{-t} dx \tag{13}$$

By making x as a subject of the formula and differentiate with respect to t from equation (12)

$$x = \left(\frac{1}{\lambda}\left((k+1)\gamma^{\theta}\beta t^{-1}\right)^{\frac{1}{\beta\theta}}\right) \tag{14}$$

$$dx = -\frac{\left((k+1)\gamma^{\theta}\beta\right)^{\frac{1}{\beta\theta}}}{\lambda\beta\theta}\, t^{-\frac{1}{\beta\theta}-1}\, dt \tag{15}$$

Substitute equation (14) and (15) into equation 13

$$E(x^r) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} w_i \int_0^{\infty}\left(\frac{1}{\lambda}\left((k+1)\gamma^{\theta}\beta t^{-1}\right)^{\frac{1}{\beta\theta}}\right)^{r-\beta\theta-1} e^{-t}\; -\frac{\left((k+1)\gamma^{\theta}\beta\right)}{\lambda\beta\theta}\, t^{-\frac{1}{\beta\theta}-1} dt$$

$$E(x^r) = -\frac{\alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta} w_i}{(k+1)\gamma^{\theta}\beta\lambda^{-\beta\theta}}\int_0^{\infty} t^{\frac{r-\beta\theta-1}{\beta\theta}}\, e^{-t} dt \tag{16}$$

r^th moment is expressed as

$$E(x^r) = -\alpha\gamma^{\frac{r}{\beta}} w_i\left((k+1)\beta\right)^{\frac{r}{\beta\theta}}\Gamma\left(1-\frac{r}{\beta\theta}\right)$$

To obtain the 1^st moment, let r =1

$$E(x) = -\alpha\gamma^{\frac{1}{\beta}} w_i\left((k+1)\beta\right)^{\frac{1}{\beta\theta}}\Gamma\left(1-\frac{1}{\beta\theta}\right)$$

To obtain the 2^nd moment, let r=2

$$E(x^2) = -\alpha\gamma^{\frac{2}{\beta}} w_i\left((k+1)\beta\right)^{\frac{2}{\beta\theta}}\Gamma\left(1-\frac{2}{\beta\theta}\right)$$

To obtain the 3^rd moment, r=3

$$E(x^3) = -\alpha\gamma^{\frac{3}{\beta}} w_i\left((k+1)\beta\right)^{\frac{3}{\beta\theta}}\Gamma\left(1-\frac{3}{\beta\theta}\right)$$

To obtain the 4^th moment, r=4

$$E(x^4) = -\alpha\gamma^{\frac{4}{\beta}} w_i\left((k+1)\beta\right)^{\frac{4}{\beta\theta}}\Gamma\left(1-\frac{4}{\beta\theta}\right)$$

Moment-Generating Function (MGF)

To compute the moment-generating function (MGF) $\boldsymbol{M_x(t)}$ for Exponentiated Fréchet-Weibull distribution

$$\boldsymbol{M_x(t) = E(e^{tx})} \tag{17}$$

$$E(e^{tx}) = \int_0^{\infty} e^{tx} f(x;\alpha,\theta,\gamma,\beta)dx \tag{18}$$

From equation (6)

$$f(x;\alpha,\theta,\gamma,\beta) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}x^{-(\beta\theta+1)} w_i e^{-(k+1)\left(\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}\right)}$$

$$M_x(t) = \int_0^{\infty} e^{tx}\,\alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}x^{-(\beta\theta+1)} w_i e^{-(k+1)\left(\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}\right)} \tag{19}$$

Substitute $\mathbf{f(x;\alpha,\theta,\gamma,\beta)}$ pdf into $\mathbf{E(e^{tx})}$

$$\boldsymbol{M_x(t)} = -\sum_{r=0}^{\infty}\frac{t^r}{r!}\alpha\gamma^{\frac{r}{\beta}}w_i\big((k+1)\beta\big)^{\frac{r}{\beta\theta}}\Gamma\left(1-\frac{r}{\beta\theta}\right)$$

$$\boldsymbol{M_x(t)} = -\sum_{r=0}^{\infty}\frac{t^r}{r!}\alpha\gamma^{\frac{r}{\beta}}w_i\big((k+1)\beta\big)^{\frac{r}{\beta\theta}}\Gamma\left(1-\frac{r}{\beta\theta}\right)$$

$$\boldsymbol{M'_x(t)} = -\sum_{r=0}^{\infty}\frac{rt^{r-1}}{r!}\alpha\gamma^{\frac{r}{\beta}}w_i\big((k+1)\beta\big)^{\frac{r}{\beta\theta}}\Gamma\left(1-\frac{r}{\beta\theta}\right)$$

$$\boldsymbol{M'_x(t)} = -\sum_{r=0}^{\infty}\frac{1t^{1-1}}{1!}\alpha\gamma^{\frac{1}{\beta}}w_i\big((k+1)\beta\big)^{\frac{1}{\beta\theta}}\Gamma\left(1-\frac{1}{\beta\theta}\right)$$

$$\boldsymbol{M'_x(t)} = \boldsymbol{M'_x(t)} = \alpha\gamma^{\frac{1}{\beta}}w_i\big((k+1)\beta\big)^{\frac{1}{\beta\theta}}\Gamma\left(1-\frac{1}{\beta\theta}\right)$$

$$\boldsymbol{M''_x(t)} = \sum_{r=0}^{\infty}\frac{r(r-1)t^{r-2}}{r!}\alpha\gamma^{\frac{r}{\beta}}w_i\big((k+1)\beta\big)^{\frac{r}{\beta\theta}}\Gamma\left(1-\frac{r}{\beta\theta}\right)$$

Now, we will substitute t=1 and r=2

$$\boldsymbol{M''_x(t)} = \sum_{r=0}^{\infty}\frac{2(2-1)t^{2-2}}{2!}\alpha\gamma^{\frac{2}{\beta}}w_i\big((k+1)\beta\big)^{\frac{2}{\beta\theta}}\Gamma\left(1-\frac{2}{\beta\theta}\right)$$

$$\boldsymbol{M''_x(t)} = \alpha\gamma^{\frac{2}{\beta}}w_i\big((k+1)\beta\big)^{\frac{2}{\beta\theta}}\Gamma\left(1-\frac{2}{\beta\theta}\right)$$

$$\boldsymbol{M'''_x(t)} = \alpha\gamma^{\frac{3}{\beta}}w_i\big((k+1)\beta\big)^{\frac{3}{\beta\theta}}\Gamma\left(1-\frac{3}{\beta\theta}\right)$$

$$\boldsymbol{M'^{v}_x(t)} = \alpha\gamma^{\frac{4}{\beta}}w_i\big((k+1)\beta\big)^{\frac{4}{\beta\theta}}\Gamma\left(1-\frac{4}{\beta\theta}\right)$$

Likelihood Function

To obtain the maximum likelihood estimators (MLEs) for the parameters α, β, λ, and θ of Exponentiated Fréchet-Weibull distribution (EFW), we need to set up the likelihood function based on the pdf and then take the partial derivatives with respect to each parameter.

Let $x_1, x_2, \ldots, x_n$ be an observed sample from the EFW distribution.

The **log-likelihood function** ℓ(α,θ,γ,β,λ) is:

$$l = \sum_{i=1}^{n} log\, f(x;\alpha,\theta,\gamma,\beta,\lambda)$$

$$f(x;\alpha,\theta,\gamma,\beta,\lambda) = \alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\, x^{-(\beta\theta+1)}e^{-\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}}\left(1-e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha-1}$$

Putting the PDF function into the likelihood-function:

$$L(x;\alpha,\theta,\gamma,\beta,\lambda) = \sum_{i=1}^{n} log\left(\alpha\theta\gamma^{\theta}\beta\lambda^{-\beta\theta}\, x^{-(\beta\theta+1)}e^{-\frac{\gamma^{\theta}}{(\lambda x)^{\beta\theta}}}\left(1-e^{-\left(\frac{\gamma}{(\lambda x)^{\beta}}\right)^{\theta}}\right)^{\alpha-1}\right) \quad (20)$$

$$L(x;\alpha,\theta,\gamma,\beta,\lambda) = nln\alpha + nln\theta + n\theta ln\gamma + nln\beta + n\beta ln\lambda - \beta(\theta+1)\sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n}(\lambda x_i)^{\beta} - \sum_{i=1}^{n}\frac{\gamma^{\theta}}{(\lambda x_i)^{\beta\theta}} + (\alpha-1)\sum_{i=1}^{n} ln\left(1-e^{-\left(\frac{\gamma}{(\lambda x_i)^{\beta}}\right)^{\theta}}\right)$$

Differentiating partially $\ell(\alpha,\theta,\gamma,\beta,\lambda)$ concerning α is:

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{\boldsymbol{n}} ln\left(1-e^{-\left(\frac{\gamma}{(\lambda x_i)^{\beta}}\right)^{\theta}}\right) \tag{21}$$

$$\frac{\partial l}{\partial \beta}=-n\theta ln\lambda-(\beta\theta+1)\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\gamma^{\theta}\theta ln(\lambda x_i)}{(\lambda x_i)^{\beta\theta}} \tag{22}$$

$$\frac{\partial l}{\partial \gamma}=\frac{n\theta}{\gamma}-\sum_{i=1}^{n}\frac{\gamma^{\theta-1}}{(\lambda x_i)^{\beta\theta}} \tag{23}$$

$$\frac{\partial l}{\partial \lambda}=-\frac{n\beta\theta}{\lambda}+\sum_{i=1}^{n}\frac{\gamma^{\theta}\beta\theta}{\lambda^{\beta\theta+1}x_i^{\beta\theta}} \tag{24}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}+nln\gamma-n\beta ln(\lambda)-n\sum_{i=1}^{n}lnx_i+\sum_{i=1}^{\boldsymbol{n}}\frac{\gamma^{\theta}ln(\gamma)-ln(\lambda x_i)}{(\lambda x_i)^{\boldsymbol{\beta\theta}}} \tag{25}$$

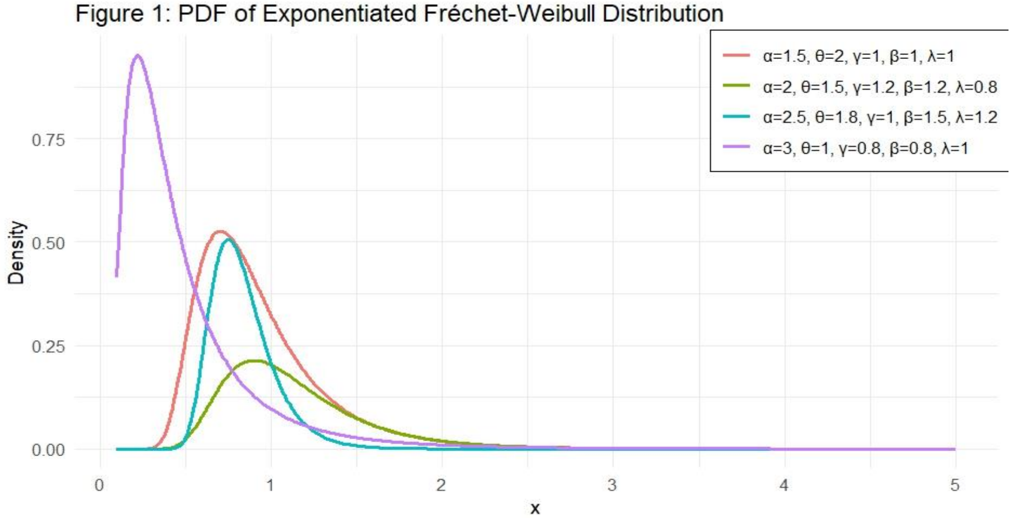
Numerical Algorithm (Optimisation)
We use **Maximum Likelihood Estimation (MLE)** via numerical optimization. We use **R** with the Quasi-Newton Raphson method) to obtain the parameters using the first dataset.

α	θ	γ	β	λ
1824.9937	1.320214	2635.7855	1.809813	36.0671

Second data

α	θ	γ	β	λ
1.4945e+02	1.8131e-01	1.3148e+02	4.0941e-01	1.7824e-03

Figure 1: PDF of Exponentiated Fréchet-Weibull Distribution



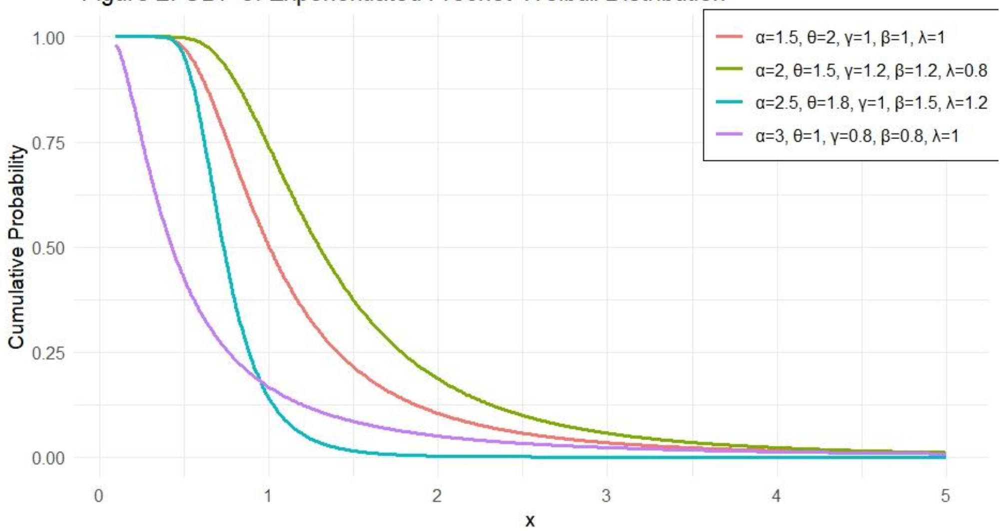


Figure 2: CDF of Exponentiated Fréchet-Weibull Distribution

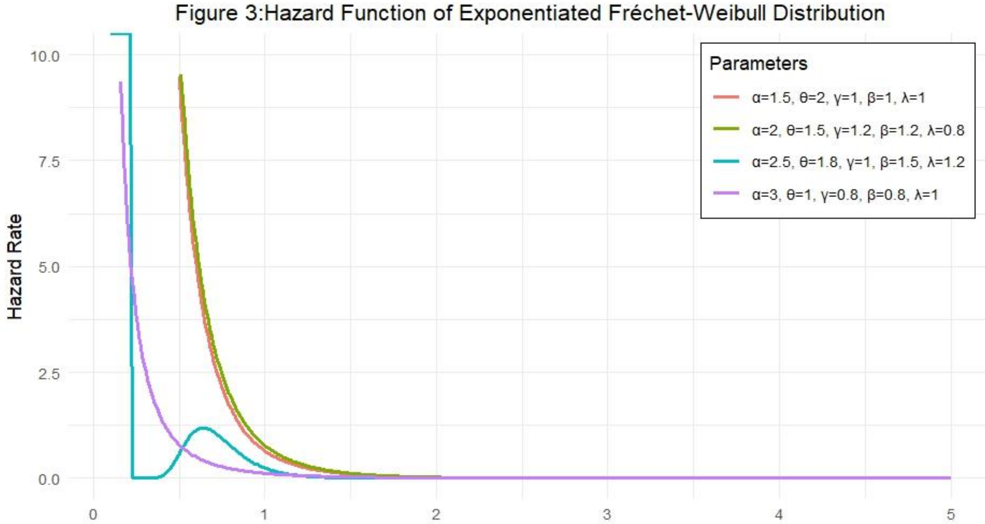


Figure 3:Hazard Function of Exponentiated Fréchet-Weibull Distribution

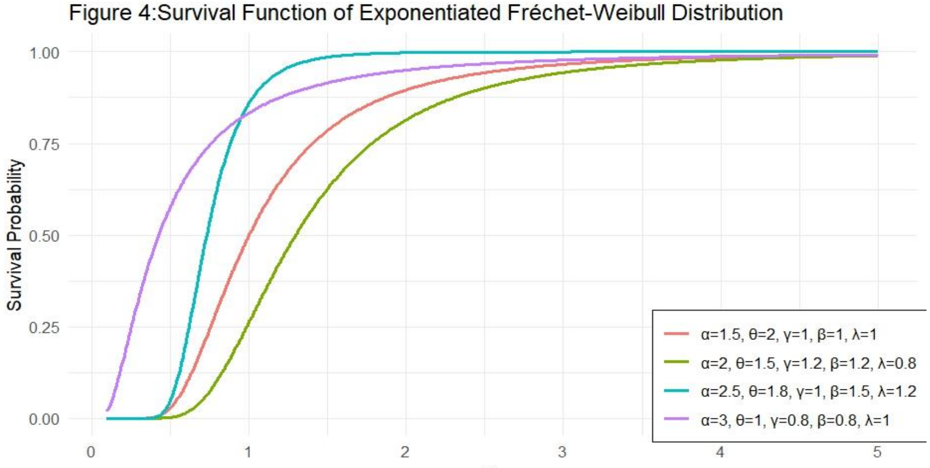


Figure 4:Survival Function of Exponentiated Fréchet-Weibull Distribution

We Consider an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba), Smith, R. L., & Naylor, J. C. (1987).: Sahai et al (2021). The breaking stress of carbon fibers dataset contains 100 positive, continuous values with right-skewed behavior, making it suitable for EF-Weibull modeling due to its ability to capture variability and heavy tails. Its flexible hazard function is ideal for analyzing material strength and failure data.

Table 1: Breaking stress of carbon fibers (in Gba)

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Source: Smith & Naylor (1987)

Table 2: Summary Statistics Data on Carbon Fibers Breaking Stress (in Gba) Data

Mean	Std.Dev	Median	Min	Max	Skew	Kurtosis	Range
2.62	1.01	2.70	0.39	5.56	0.36	0.04	5.17

Source: Authors computation R statistics

This table 2 indicates key descriptive statistics such as mean, standard deviation, median, minimum, maximum, skewness, kurtosis, and range**.**

Table 3: Model comparison using **Loglikelihood and Information Criterion of breaking stress Data**

Distribution	LogLikelihood	AIC	BIC
Exponential	-413.0733	828.1467	830.7518
Exp-Exponential	-403.9095	811.8190	817.0293
Weibull	-268.3168	540.6337	545.8440
Fréchet	-208.8769	421.7538	426.9641
Exp-Weibull	-260.0399	526.0799	533.8954
Exp-Fréchet	-205.8048	417.6097	425.4252
EFW	-30.9813	69.9626	80.38328

Source: Authors computation R statistics

Table 3 presents a comparative analysis of selected probability distributions used to model the breaking stress of carbon fibers, based on log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). A higher log-likelihood value and lower AIC/BIC scores indicate a better model fit. Among the models considered, the Exponential distribution demonstrated the poorest performance, while the Exponentiated Fréchet-Weibull (EFW) distribution provided the best fit. The EFW model achieved the highest log-likelihood (-30.9813) and the lowest AIC (69.9626) and BIC (80.3833), identifying it as the most optimal model. Although models such as the Weibull and Fréchet showed moderate performance, they were less effective compared to EFW. The AIC and BIC criteria penalize model complexity, ensuring a balance between goodness-of-fit and parsimony. Overall, the findings support the EFW distribution as the most suitable model for characterizing the breaking stress data.

Table 4: Goodness of fit Criterion of breaking stress Data

Distribution	KS_Statistic	AD_Statistic	CM_Statistic
Exponential	0.6922	31.0823	20.8056
Frechet	0.4498	32.3462	6.9945
Weibull	0.6751	113.5178	19.5529
Exp_Exponential	0.6762	122.8361	19.9059
Exp_Frechet	0.4847	38.1370	8.3292
Exp_Weibull	0.6578	105.5068	18.5685
EFW	0.1956	15.9627	3.1824

Source: Authors computation R statistics

Table 4 presents the goodness-of-fit evaluation of various probability distributions applied to the breaking stress data, based on the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CM) statistics. The Exponentiated Fréchet–Weibull (EFW) distribution exhibits the lowest KS (0.1956), AD (15.9627), and CM (3.1824) values, indicating the best overall fit. In contrast, the Exponential distribution shows the poorest performance, with the highest KS (0.6922), AD (31.0823), and CM (20.8056) statistics. The Fréchet and Exponentiated Fréchet (Exp-Fréchet) distributions perform moderately—better than the Exponential model but less effectively than the EFW. Meanwhile, the Exponentiated Weibull (Exp-Weibull), Weibull, and Exponentiated Exponential (Exp-Exponential) distributions exhibit relatively high AD and CM values, suggesting poorer fits. The AD statistic is particularly sensitive to deviations in the distribution tails, while the CM statistic measures overall deviation. These results affirm that the EFW distribution provides the most appropriate fit for the breaking stress data.

Table 5: Data represents a COVID-19 mortality rate

8.826,6.105,10.383,7.267,13.220,6.015,10.855,6.122,10.685,10.035,5.242,7.630,14.604,7.903, 6.327,9.391,14.962,4.730,3.215,16.498,11.665,9.284,12.878,6.656,3.440,5.854,8.813,10.043,7 .260,5.985,4.424,4.344,5.143,9.935,7.840,9.550,6.968,6.370,3.537,3.286,10.158,8.108,6.697,7 .151,6.560,2.988,3.336,6.814,8.325,7.854,8.551,3.228,3.499,3.751,7.486,6.625,6.140,4.909,4. 661,1.867,2.838,5.392,12.042,8.696,6.412,3.395,1.815,3.327,5.406,6.182,4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120,3.922,3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027,2.352, 1.205, 2.077, 3.778,3.218,2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923

Source: Mortality rate of Mexico

Table 5 illustrates the Mexico COVID-19 mortality rate of 108 days, recorded from 4 March to 20 July 2020. The mortality rate dataset from Mexico spans 108 days with non-negative, continuous values showing irregular spikes and tail behavior. The EF-Weibull distribution is appropriate here due to its flexibility in modeling epidemic mortality patterns with complex hazard structures.

Table 6: Summary Statistics of COVID-19 mortality rate data that belongs to Mexico within 108 days

Mean	Sd	Median	Min	Max	Skew	Kurtosis	Range
5.76	3.25	5.19	1.04	16.5	0.97	0.61	15.46

Source: Authors computation R statistics

Table 6 describes key statistics such as mean, standard deviation, median, minimum, maximum, skewness, kurtosis, and range**.**

Table 7: Model comparison using Log likelihood and Information Criterion of Mexico COVID-19 Data

Distribution	Log Likelihood	AIC	BIC
Exponential	-1055.8850	2113.7701	2116.4522
Exp-Exponential	-1045.6677	2095.3354	2100.6997
Weibull	-580.3746	1164.7492	1170.1134
Fréchet	-331.0231	666.0461	671.4104
Exp-Weibull	-570.4822	1146.9644	1155.0108
Exp-Fréchet	-332.8664	671.7328	679.7792
EFW	-4.8897	17.7794	28.5080

Table 4 presents the goodness-of-fit evaluation of various probability distributions applied to the breaking stress Data, based on the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CM) statistics. The Exponentiated Fréchet–Weibull (EFW) distribution exhibits the lowest KS (0.1956), AD (15.9627), and CM (3.1824) values, indicating the best overall fit. In contrast, the Exponential distribution shows the poorest perform ance, with the highest KS (0.6922), AD (31.0823), and CM (20.8056) statistics. The Fréchet and Exponentiated Fréchet (Exp-Fréchet) distributions perform moderately better than the Exponential model but less effective than the EFW. Meanwhile, the Exponentiated Weibull (Exp-Weibull), Weibull, and Exponentiated Exponential (Exp-Ex ponential) distributions exhibit relatively high AD and CM values, suggesting poorer fits. The AD statistic is partic ularly sensitive to deviations in the distribution tails, while the CM statistic measures overall deviation. These results affirm that the EFW distribution provides the most appropriate fit for the breaking stress data

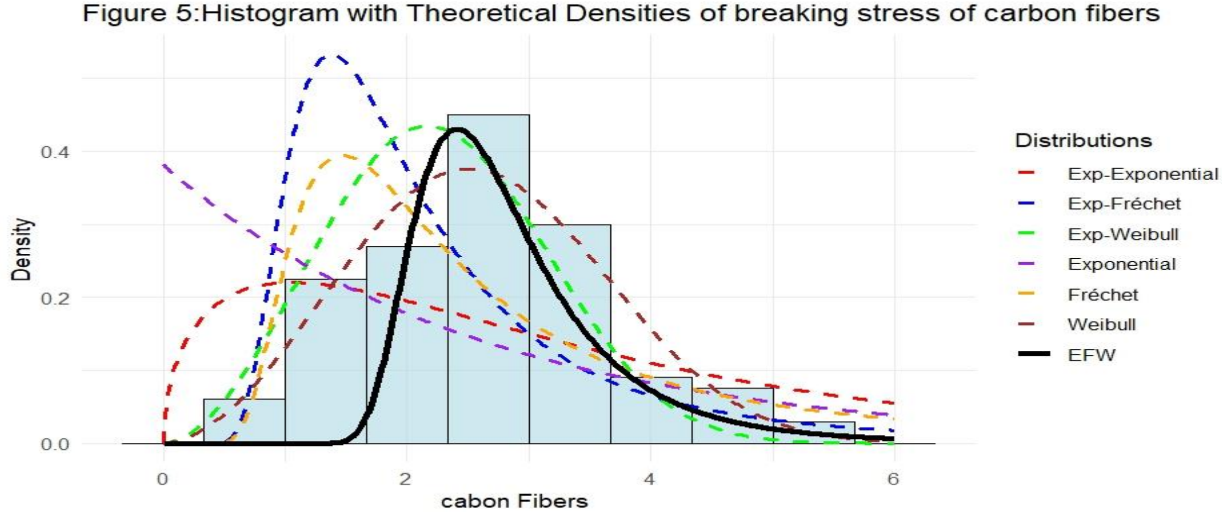
Table 8: Goodness of fit Criterion of COVID-19 mortality rate data that belongs to Mexico within 108 days.

Distribution	KS_Statistic	AD_Statistic	CM_Statistic
Exponential	0.8342	366.3142	31.2414
Frechet	0.5898	68.4413	14.4674
Weibull	0.8149	297.2201	30.1016
Exp_Exponential	0.8214	356.3606	30.8391
Exp_Frechet	0.6248	81.28323	16.5883
Exp_Weibull	0.8006	287.3742	29.6044
EFW	0.1756	17.7917	3.57210

Table 8 presents the Goodness-of-Fit (GoF) criteria for different distributions fitted to the COVID-19 mortality rate data over 108 days in Mexico. The Kolmogorov-Smirnov (KS), Anderson-Darling (AD), and Cramér-von Mises (CM) statistics are used to assess how well each distribution fits the data. The Exponentiated Fréchet-Weibull (EFW) distribution has the lowest KS (0.1756), AD (17.7917), and CM (3.5721) statistics, indicating the best fit among all models. The Exponential distribution shows the worst fit, with the highest KS (0.8342), AD (366.3142), and CM (31.2414) statistics, suggesting significant deviation from the data.

The Fréchet and Exp-Fréchet distributions perform better than Weibull but still do not match EFW's fit. The Exp-Weibull and Exp-Exponential distributions improve over Weibull and Exponential but remain less effective than EFW. Lower KS, AD, and CM values indicate a better model fit, confirming that the EFW distribution provides the most accurate representation of COVID-19 mortality rate data.

Figure 5:Histogram with Theoretical Densities of breaking stress of carbon fibers



This plot in figure 5 presents a histogram with theoretical density curves overlaid, comparing different probability distributions fitted to a dataset. The histogram, represented by light blue bars, shows the empirical distribution of the data. The black line represents the Exponentiated Fréchet-Weibull (EFW) distribution, which closely follows the shape of the histogram. This suggests that the EFW distribution provides the best fit among all the tested models, as it captures both the peak and the tail behavior effectively. Other distributions, such as the Exponential (purple), Exp-Exponential (red), and Exp-Weibull (green), deviate significantly from the histogram, particularly at the peak and the tail. The Exponential model performs poorly, as it fails to capture the skewness of the data. The Fréchet (orange) and Exp-Fréchet (blue) distributions exhibit a moderate fit but still struggle to align with the data, especially at the peak and right tail. The Weibull (brown) distribution fits better than the Exponential model but does not achieve the level of accuracy that the EFW model demonstrates.

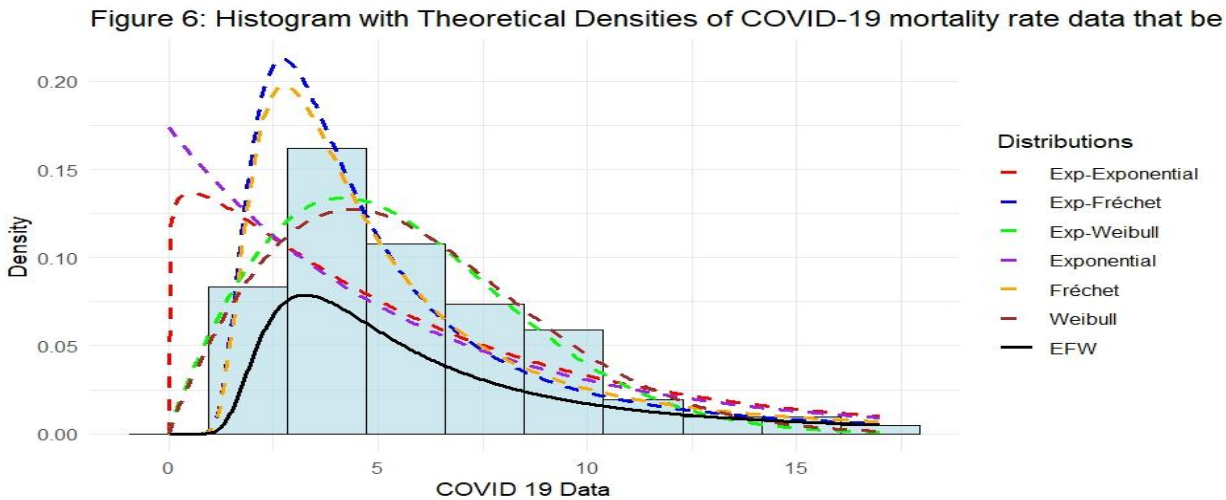


Figure 6 presents a histogram of the observed data overlaid with the theoretical density functions of various probability distributions. The histogram, displayed in light blue bars, illustrates the empirical distribution, while the dashed curves represent different fitted models. The solid black curve corresponds to the Exponentiated Fréchet-Weibull (EFW) distribution. The x-axis denotes the observed variable (e.g., breaking stress or COVID-19 mortality rate), and the y-axis represents the probability density.

The EFW distribution demonstrates the best fit, as its curve closely follows the empirical histogram, effectively capturing both the central peak and the long right tail. In contrast, distributions such as the Exp-Exponential (red) and Exp-Fréchet (blue) deviate significantly, with their peaks misaligned and poor representation of the data's spread. The Weibull (brown) and Fréchet (orange) models offer moderate fits but still fail to capture the tail behavior accurately. The Exponential distribution (purple) performs the worst, as it is unable to accommodate the skewness and heavy-tailed nature of the data.

CONCLUSION

The Exponentiated Fréchet–Weibull (EF-Weibull) distribution was developed by integrating the Weibull cumulative distribution function (CDF) into the Exponentiated Fréchet-G (EF-G) family, resulting in a highly flexible statistical model capable of accommodating extreme values and heavy-tailed data. The derivation of its probability density function (PDF), CDF, quantile function, and hazard function illustrates the model's ability to capture a wide range of data behaviors. Moment-based analyses, including the r-th moment and moment-generating function (MGF), were employed to explore the distribution's theoretical properties. Parameter estimation was performed using the maximum likelihood estimation (MLE) method, reinforcing the EF-Weibull distribution's practical applicability in real-world contexts. A comparative analysis was carried out to evaluate the performance of the EF-Weibull distribution relative to several alternative models, including the Exponentiated Fréchet (Exp-Fréchet), Exponentiated Weibull (Exp-Weibull), Exponentiated Exponential (Exp-Exponential), Fréchet, and Weibull distributions. Each model was fitted to empirical data using MLE, and their goodness-of-fit was assessed using log-likelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov–Smirnov (KS) test, Anderson–Darling (AD) test, and Cramér–von Mises (CM) test. The EF-Weibull distribution achieved the highest log-likelihood and the lowest AIC and BIC values, indicating the best fit among the models. It also recorded the lowest KS, AD, and CM statistics, further confirming its superior alignment with empirical data. While the Fréchet and Exp-Fréchet models performed relatively well, they lacked the flexibility of the EF-Weibull in capturing data variability. The Weibull and Exp-Weibull distributions showed moderate performance but were limited in modeling tail behavior. The Exp-Exponential distribution was the least effective, failing to capture extreme values adequately.Graphical evaluations including histogram and theoretical density plots, hazard function plots, and empirical CDF comparisons provided additional support for the EF-Weibull distribution's superior performance. The histogram and density plots demonstrated a strong visual alignment of the EF-Weibull model with observed data, especially in the tail regions. The hazard function analysis showcased the distribution's adaptability for reliability modeling, while empirical CDF comparisons further affirmed its excellent fit to real-world data.

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