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Original Research Article

DEVELOPMENT OF GOMPERTZ GENERALISED WEIBULL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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This study was a collaborative effort among all authors. Each author reviewed and approved the final version of the manuscript for publication.

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ABSTRACT
This paper introduces the Gompertz Generalised Weibull (GGWeibull) distribution, a novel and flexible statistical model obtained by applying the Gompertz generator to the Weibull baseline distribution. The GGWeibull enhances modelling capabilities by accommodating a wide range of hazard shapes and tail behaviours, making it particularly well-suited for reliability and engineering data. We derive and explore its key statistical properties, including the probability density function, cumulative distribution function, survival function, hazard function, and moments. Parameter estimation is conducted via the maximum likelihood method, and the model's performance is rigorously assessed through goodness-of-fit measures, including the log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC). Application to the breaking stress data of carbon fibres demonstrates the clear superiority of the GGWeibull distribution over traditional models such as the Weibull, Gompertz, Exponentiated Gompertz, and Exponentiated Weibull distributions. The GGWeibull achieves the highest log-likelihood and the lowest AIC and BIC, indicating the best balance between model fit and complexity. Visual comparisons using histograms and theoretical density plots further confirm its superior fit to empirical data. We recommend the GGWeibull distribution as a robust and flexible tool for modelling failure times, material strength, and other engineering phenomena, and encourage future research to extend its application to broader fields such as biomedical survival analysis and environmental risk modelling.

Keywords: Reliability analysis, Hazard function, Maximum likelihood, Application, Properties

INTRODUCTION

In recent years, the development of flexible probability distributions has become a major focus in statistical modelling, driven by the need to better capture the complexities of real-world data across fields such as reliability analysis, survival analysis, hydrology, economics, and medicine. Among these innovations, the Gompertz Generalized Weibull Distribution (GGWD) emerges as a novel distribution derived from the Gompertz-G family, using the Weibull distribution as the baseline. This new model enriches the flexibility of the classical Weibull distribution by incorporating the shape-enhancing mechanism of the Gompertz-G framework, allowing it to handle various data behaviors such as increasing, decreasing, bathtub, and unimodal hazard rates. The Gompertz-G family, proposed by Morad A. et al (2017), has been widely used to generalize well-known distributions by embedding the Gompertz generator, which introduces an additional parameter controlling the tail and hazard behaviors. When applied to the Weibull baseline, the resulting GGWD offers enhanced modeling capability over the traditional Weibull and Gompertz-Weibull models. In the broader landscape of distributional developments, several authors have introduced and extended families that aim to achieve better goodness-of-fit, flexibility, and interpretability. Recent examples include the Exponentiated Weibull (Mudholkar and Srivastava, 1993), Kumaraswamy Weibull (Cordeiro and de Castro, 2011), Weibull-G (Bourguignon et al., 2014), Beta Weibull (Famoye et al., 2005), Exponentiated Generalized Weibull (Cordeiro et al., 2013), Weibull Poisson (Morais and Barreto-Souza, 2011), Marshall-Olkin Weibull (Li and Peng, 2011), Weibull Lindley (Zakerzadeh and Dolati, 2009), Weibull Burr XII (Cordeiro et al., 2015), Exponential Generalized Weibull (Alizadeh et al., 2015), Gompertz Weibull (Zografos and Balakrishnan, 2009), Exponentiated Exponential Weibull (Sarhan et al., 2011), Truncated Weibull Lomax (Bakouch et al., 2014), Transmuted Weibull (Aryal and Tsokos, 2009), Weibull Inverse Gaussian (Hossain et al., 2012), Gamma Weibull (Hassanein and Al-Shomrani, 2015), Generalized Gamma Weibull (Nadarajah and Kotz, 2006), Fréchet Weibull (El-Gohary et al., 2013), Weibull Pareto (Bourguignon et al., 2016), and Weibull Power Series (Barreto-Souza et al., 2011). These distributions have been used to model diverse types of lifetime, reliability, and survival data, often outperforming classical models in terms of goodness-of-fit and providing deeper insights into the stochastic behaviour of complex systems. By situating the GGWD within this rich framework, this work aims to extend the boundaries of applicable models, offering a versatile tool for practitioners and researchers across many applied disciplines.

METHODOLOGY

The Gompertz-G Family of Distributions

This paper presents the cumulative distribution function (CDF), probability density function (PDF), and quantile function of the GGFD, alongside its properties and potential applications.
The Gompertz-G family of distributions is defined by its cumulative distribution function:

$$F(x) = 1 - e^{\frac{\theta}{\gamma}\left[1-(1-G(x))^{-\gamma}\right]} \qquad \gamma \text{ and } \theta>0 \tag{1}$$

Where: θ >0 is the scale parameter,

- γ >0 is the shape parameter,
- G(x) is the CDF of the baseline distribution.

The corresponding PDF is given by:

$$f(x) = \theta g(x)\left(1 - G(x)\right)^{-\gamma-1} e^{\frac{\theta}{\gamma}\left[1-(1-G(x))^{-\gamma}\right]} \tag{2}$$

Where g(x) is the PDF of the baseline distribution.

The Weibull Distribution

The Weibull distribution is widely used in modelling extreme values. Its CDF, PDF, and quantile function are given as:

CDF:

$G_W(x) = 1 - e^{-(\lambda x)^{\beta}}$Where λ, β > 0 is the scale parameter and $\theta > 0$ is the shape parameter.

PDF:

$$g_{EL}(x) = \beta\lambda x^{\beta-1} e^{-(\lambda x)^{\beta}} \qquad X > 0.\ \alpha, \lambda\ and\ \beta > 0 \tag{3}$$

Quantile Function:

$$Q_{EL}(x) = \frac{1}{\lambda}\left[\left(-\ln(1-p)\right)^{\frac{1}{\beta}}\right] \tag{4}$$

The Gompertz Generalised Weibull Distribution

Using the Gompertz-G framework, the GGFD is derived with the Weibull distribution as the baseline. Substituting G(x) and g(x) from the Weibull distribution into the Gompertz-G formulas, we obtain:

CDF:

$$F(x) = 1 - e^{\frac{\theta}{\gamma}\left[1-\left(e^{-(\lambda x)^{\beta}}\right)^{-\gamma}\right]} \tag{5}$$

$$F(x) = 1 - e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]} \tag{6}$$

PDF:

$$f(x) = \theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)} \quad (7)$$

Using Taylor series expansion

$$e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)} = \sum_{i=o}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}}{i}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)^{i}$$

$$\sum_{i=o}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}}{i}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)^{i} = \sum_{i=o}^{\infty}\frac{1}{i}\left(\frac{\theta}{\gamma}\right)^{i}\sum_{j=0}^{\infty}(-1)^{j}\binom{i}{j}\left(e^{\gamma(\lambda x)^{\beta}}\right)^{j}$$

$$e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)} = \frac{1}{i}\sum_{i=o}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j}\left(e^{\gamma(\lambda x)^{\beta}}\right)^{j}$$

$$f(x) = \theta\beta\lambda^{\beta}x^{\beta-1}\left(e^{\gamma(\lambda x)^{\beta}}\right)\frac{1}{i}\sum_{i=o}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j}\left(e^{\gamma(\lambda x)^{\beta}}\right)^{j}$$

$$f(x) = \theta\beta\lambda^{\beta}x^{\beta-1}\sum_{i=o}^{\infty}\sum_{j=1}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}}{i}(-1)^{j}\binom{i}{j}\left(e^{\gamma(J+1)(\lambda x)^{\beta}}\right)$$

$let\ t = (\lambda x)^{\beta}$

And

$$w_i = \sum_{i=o}^{\infty}\sum_{j=1}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}}{i}(-1)^{j}\binom{i}{j}$$

$$f(x) = \theta\beta\lambda^{\beta}x^{\beta-1}\,w_i\left((e^{-t})^{-\gamma(J+1)}\right) \quad (8)$$

Quantile Function:

$$Q(p) = \beta\left[-\frac{1}{\lambda}\ln\left(1-\left(\frac{\gamma}{\theta}\ln\left(\frac{1}{1-p}\right)\right)^{-\frac{1}{\gamma}}\right)\right]^{\frac{1}{\alpha}} \quad (9)$$

Survival Function of Gompertz Generalized Exponential Lomax Distribution

$$s(x) = 1 - F(x)$$

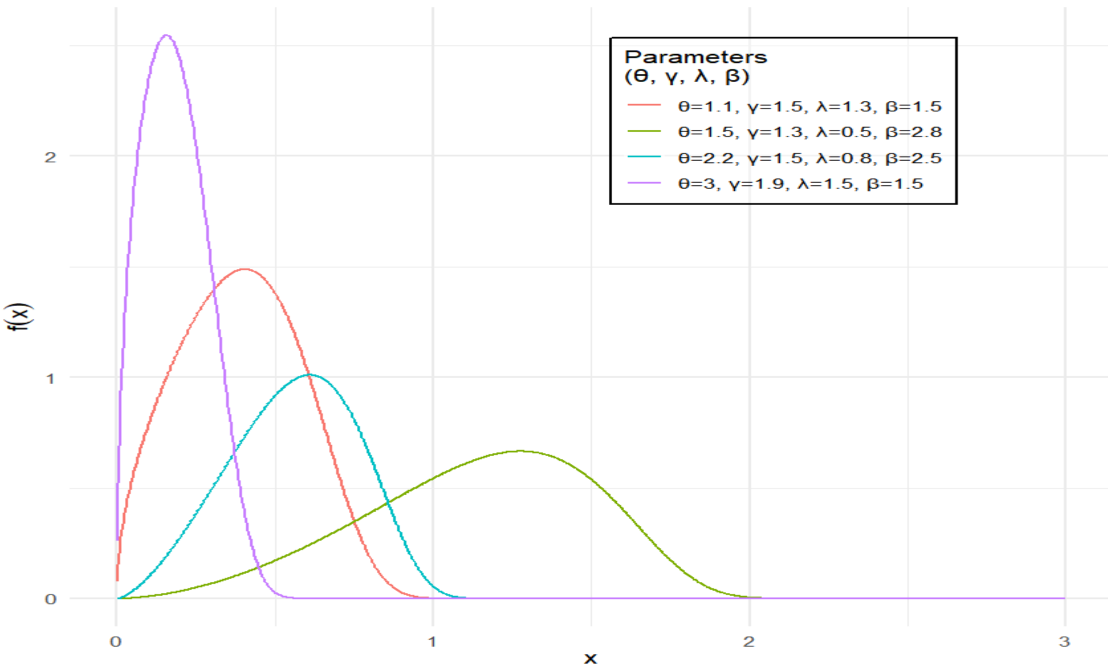
$$s(x) = 1 - \left(1 - e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right)$$

$$s(x) = e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]} \quad (10)$$

Hazard Function of Gompertz Generalized Exponential Lomax Distribution

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$$

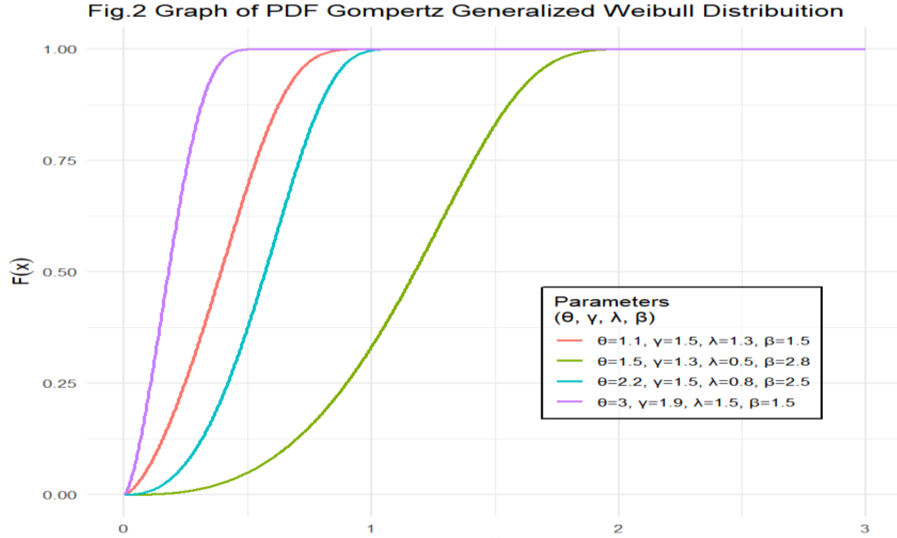
Fig.1 Graph of PDF Gompertz Generalized Weibull Distribuition



$$h(x) = \frac{\theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}}{e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}}$$

$$h(x) = \theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}} \quad (11)$$

Fig.2 Graph of PDF Gompertz Generalized Weibull Distribuition



Moments

The r-th moment of the EGG-EL distribution is given by:

$u_r' = E(x^r) = \int_0^\infty x^r f(x) dx$

$f(x) = \theta\beta\lambda^\beta x^{\beta-1}\, w_i e^{\gamma(J+1)t}$

$E(x^r) = \int_0^\infty x^r \theta\beta\lambda^\beta x^{\beta-1}\, w_i e^{\gamma(J+1)t} dx$

$E(x^r) = \theta\beta\lambda^\beta w_i \int_0^\infty x^{r+\beta-1} e^{\gamma(J+1)t} dx$

$since\ t = (\lambda x)^\beta$

$x = \frac{1}{\lambda} t^{\frac{1}{\beta}}$

$\frac{dx}{dt} = \frac{1}{\lambda\beta} t^{\frac{1}{\beta}-1}$

$dx = \frac{1}{\lambda\beta} t^{\frac{1}{\beta}-1} dt$

By switching to z form

$E(x^r) = \int_0^\infty E(x^r) = \theta\beta\lambda^\beta w_i \int_0^\infty \left(\frac{1}{\lambda} t^{\frac{1}{\beta}}\right)^{r+\beta-1} e^{\gamma(J+1)t} \frac{1}{\lambda\beta} t^{\frac{1}{\beta}-1} dt$

$t^{\frac{r+\beta-1}{\beta}} \times t^{\frac{1}{\beta}-1} = t^{\frac{r+\beta-1}{\beta}+\frac{1-\beta}{\beta}} = t^{\frac{r}{\beta}}$

$\lambda^\beta \times \frac{1}{\lambda^{r+\beta-1}} \times \frac{1}{\lambda} = \lambda^{-r}$

$E(x^r) = \theta\lambda^{-r} w_i \int_0^\infty t^{\frac{r}{\beta}} e^{\gamma(J+1)t} dt$

$let\ \frac{r}{\beta} = s$

Therefore

$\int_0^\infty t^s e^{-at} dz = \frac{\Gamma(s+1)}{a^{s+1}}$

$Substitute\ \ \frac{r}{\beta} = s \quad and \quad where\ a\ =\ \gamma(j+1)$

The final expression for rth Moment

$$E(x^r) = \theta\lambda^{-r} w_i \frac{\Gamma\left(\frac{r}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{r}{\beta}+1}} \qquad (25)$$

$1^{st}\ moment =\ E(x) = \mu_1' = \theta\lambda^{-1} w_i \frac{\Gamma\left(\frac{1}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{1}{\beta}+1}}$

$2^{nd}\ moment = E(x^2) = \mu_2' = \theta\lambda^{-2} w_i \frac{\Gamma\left(\frac{2}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{2}{\beta}+1}}$

$3^{rd}\ moment = E(x^3) = \mu_2' = \theta\lambda^{-3} w_i \frac{\Gamma\left(\frac{3}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{3}{\beta}+1}}$

$4^{th}\ moment = E(x^4) = \mu_2' = \theta\lambda^{-4} w_i \frac{\Gamma\left(\frac{4}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{2}{\beta}+1}}$

Moment Generating Function (MGF)
The MGF of the EGG-EL distribution is expressed as:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$
$$f(x) = \theta\beta\lambda^\beta x^{\beta-1}\, w_i e^{\gamma(J+1)t}$$
$$M_x(t) = \int_0^\infty e^{tx}\theta\beta\lambda^\beta x^{\beta-1}\, w_i e^{\gamma(J+1)t} dx$$
$$M_x(t) = \theta\beta\lambda^\beta w_i \int_0^\infty e^{tx}. x^{\beta-1} e^{\gamma(J+1)t} dx$$
$$since\ z = (\lambda x)^\beta$$
$$x = \frac{1}{\lambda} z^{\frac{1}{\beta}}$$
$$\frac{dx}{dz} = \frac{1}{\lambda\beta} z^{\frac{1}{\beta}-1}$$
$$dx = \frac{1}{\lambda\beta} z^{\frac{1}{\beta}-1} dz$$
$$M_x(t) = \theta\beta\lambda^\beta w_i \int_0^\infty e^{t\left(\frac{1}{\lambda}z^{\frac{1}{\beta}}\right)}.\left(\frac{1}{\lambda}z^{\frac{1}{\beta}}\right)^{\beta-1} e^{\gamma(J+1)z}\frac{1}{\lambda\beta} z^{\frac{1}{\beta}-1} dz$$
$$z^{\frac{\beta-1}{\beta}}. z^{\frac{1}{\beta}-1} = z^0 = 1$$
$$\left(\frac{1}{\lambda}\right)^{\beta-1}. \frac{1}{\lambda}. \lambda^\beta = 1$$
$$M_x(t) = \theta\lambda^\beta w_i \int_0^\infty e^{t\left(\frac{1}{\lambda}z^{\frac{1}{\beta}}\right)} e^{\gamma(J+1)z} dz$$
According to Taylor series expansion
$$e^{t\left(\frac{1}{\lambda}(z)^{\frac{1}{\beta}}\right)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}.\left(\frac{1}{\lambda}\right)^n (z)^{\frac{n}{\beta}}$$
$$M_x(t) = \theta\beta w_i \int_0^\infty \sum_{n=0}^{\infty} \frac{t^n}{n!}.\left(\frac{1}{\lambda}\right)^n (z)^{\frac{n}{\beta}}. e^{\gamma(J+1)z} dz$$
$$M_x(t) = \theta w_i \sum_{n=0}^{\infty} \frac{t^n}{n!}.\left(\frac{1}{\lambda}\right)^n \int_0^\infty z^{\frac{n}{\beta}}. e^{\gamma(J+1)z} dz$$
$$M_x(t) = \theta w_i \sum_{n=0}^{\infty} \frac{t^n}{n!}.\left(\frac{1}{\lambda}\right)^n \int_0^\infty z^{\frac{n}{\beta}}. e^{\gamma(J+1)z} dz$$
$$M_x(t) = \frac{\theta}{\lambda^n} w_i \frac{\Gamma\left(\frac{n}{\beta}+1\right)}{[\gamma(j+1)]^{\frac{n}{\beta}+1}} \sum_{n=0}^{\infty} \frac{(t)^n}{n!}$$

Parameter Estimation
The parameters of the EGG-F distribution are estimated using the maximum likelihood estimation (MLE) method. Let x_1, x2,…,x_n,be a random sample from the GG-EL distribution. The likelihood function L (θ,λ,α,β,γ) for a sample $x_1, x_2, \ldots, x_n$ is the product of the PDFs of each observation:

$$L(x;\theta,\beta,\lambda,\gamma) = \prod_{i=1}^{n} f(x;\theta,\beta,\lambda,\gamma)$$
$$f(x;\alpha,\theta,\beta,\lambda) = \theta\beta\lambda^\beta x^{\beta-1} e^{\gamma(\lambda x)^\beta} e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^\beta}\right)}$$
$$L(x;\alpha,\theta,\beta,\lambda,\gamma) = \prod_{i=1}^{n}\left(\theta\beta\lambda^\beta x^{\beta-1} e^{\gamma(\lambda x)^\beta} e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^\beta}\right)}\right)$$
Now, take the natural logarithm to get the log-likelihood function:

$$L(\theta,\lambda,\alpha,\beta,\gamma)=\sum_{i=1}^{n}\left(log\theta+log\lambda+log\alpha-log\beta+(\alpha-1)log\left(\frac{x_i+\beta}{\beta}\right)+\lambda\gamma\left(\frac{x_i+\beta}{\beta}\right)^{\alpha}+\frac{\theta}{\gamma}\left(1-e^{-\lambda\left(\frac{x_i+\beta}{\beta}\right)^{\alpha}}\right)\right)$$

$$l(\theta,\lambda,\alpha,\beta,\gamma)=nlog\theta+nlog\beta+n\beta log+(\beta-1)\sum_{i=1}^{n}log(x_i)+\gamma\sum_{i=1}^{n}(\lambda x_i)^{\beta}+\frac{\theta}{\gamma}\sum_{i=1}^{n}\left(1-e^{\gamma(\lambda x_i)^{\beta}}\right)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+nlog\lambda+\sum_{i=1}^{n}log(x_i)+\gamma\sum_{i=1}^{n}(\lambda x_i)^{\beta}log(\lambda x_i)^{\beta}-\theta\sum_{i=1}^{n}e^{\gamma(\lambda x_i)^{\beta}}(\lambda x_i)^{\beta}x_i log$$

$$\frac{\partial\ell}{\partial\gamma}=\sum_{i=1}^{n}(\lambda x_i)^{\beta}-\frac{\theta}{\gamma^2}\sum_{i=1}^{n}\left(1-e^{\gamma(\lambda x_i)^{\beta}}\right)-\frac{\theta}{\gamma}\sum_{i=1}^{n}\left(e^{\gamma(\lambda x_i)^{\beta}}\right)(\lambda x_i)^{\beta}$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n\beta}{\lambda}+\gamma\beta\sum_{i=1}^{n}(\lambda x_i)^{\beta}x_i-\theta\beta\sum_{i=1}^{n}e^{\gamma(\lambda x_i)^{\beta}}(\lambda x_i)^{\beta-1}x_i$$

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-\frac{1}{\gamma}\sum_{i=1}^{n}\left(1-e^{\gamma(\lambda x_i)^{\beta}}\right)$$

Order of Statistics

The density $f_{i:n}(x)$ of the ith order statistic for *i*=1,…n, from independent identically distributed random variable $Y_1,\ldots,\ Y_n$ is given by

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}F(x)^{i-1}\{1-F(x)\}^{n-i} \qquad (26)$$

CDF:

$$F(x)=1-e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}$$

PDF:

$$f(x)=\theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}$$

Substitute the CDF and pdf in equation (32)

$$f_{i:n}(x)=\frac{\theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}}{B(i,n-i+1)}\left(1-e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right)^{i-1}\left[1-\left(1-e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right)\right]^{n-i}$$

$$f_{i:n}(x)=\frac{\theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}}{B(i,n-i+1)}\left(1-e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right)^{i-1}\left[e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right]^{n-i}$$

$$e^{\frac{\theta}{\gamma}\left(1-e^{\gamma(\lambda x)^{\beta}}\right)}=\sum_{g=0}^{\infty}\sum_{h=0}^{\infty}(-1)^{g}\left(\frac{\theta}{\gamma}\right)^{g}\binom{g}{h}\left(e^{\gamma h(\lambda x)^{\beta}}\right)$$

$$\left(1-e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right)^{i-1}=\frac{1}{k!}\left(\frac{\theta}{\gamma}\right)^{k}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}\binom{k}{l}\binom{i-1}{j}\left(e^{\gamma(\lambda x)^{\beta}}\right)^{l}$$

$$\left[e^{\frac{\theta}{\gamma}\left[1-e^{\gamma(\lambda x)^{\beta}}\right]}\right]^{n-i}=\frac{1}{m!}\left((n-1)\left(\frac{\theta}{\gamma}\right)\right)^{m}\sum_{m=o}^{\infty}\sum_{r=0}^{\infty}(-1)^{r}\binom{m}{r}e^{\gamma r(\lambda x)^{\beta}}$$

$$f_{i:n}(x)=\frac{\theta\beta\lambda^{\beta}x^{\beta-1}e^{\gamma(\lambda x)^{\beta}}}{B(i,n-i+1)}\frac{1}{t!}\left(\frac{\theta}{\gamma}\right)^{t}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty}(-1)^{g}\binom{g}{h}\left(e^{\gamma h(\lambda x)^{\beta}}\right)\frac{1}{k!}\left(\frac{\theta}{\gamma}\right)^{k}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}\binom{k}{l}\binom{i-1}{j}\left(e^{l\gamma(\lambda x)^{\beta}}\right)\frac{1}{m!}$$
$$\times\left((n-1)\left(\frac{\theta}{\gamma}\right)\right)^{m}\sum_{m=o}^{\infty}\sum_{r=0}^{\infty}(-1)^{r}\binom{m}{r}e^{\gamma r(\lambda x)^{\beta}}$$

$$f_{i:n}(x)=\frac{\theta\beta\lambda^{\beta}x^{\beta-1}}{B(i,n-i+1)}\frac{1}{t!}\left(\frac{\theta}{\gamma}\right)^{t}\frac{1}{k!}\left(\frac{\theta}{\gamma}\right)^{k}\frac{1}{m!}$$
$$\times\left((n-1)\left(\frac{\theta}{\gamma}\right)\right)^{m}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=o}^{\infty}\sum_{r=0}^{\infty}(-1)^{g+j+l+r}\binom{g}{h}\binom{k}{l}\binom{i-1}{j}\binom{m}{r}e^{\gamma(1+h+l+r)(\lambda x)^{\beta}}$$

$$let\ z_i = \sum_{g=0}^{\infty}\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=o}^{\infty}\sum_{r=0}^{\infty}(-1)^{g+j+l+r}\binom{g}{h}\binom{k}{l}\binom{i-1}{j}\binom{m}{r}$$

Therefore

$$f_{i:n}(x) = \frac{\theta\beta\lambda^{\beta}x^{\beta-1}}{B(i,n-i+1)}\frac{1}{t!}\left(\frac{\theta}{\gamma}\right)^{t}\frac{1}{k!}\left(\frac{\theta}{\gamma}\right)^{k}\frac{1}{m!}\times\left((n-1)\left(\frac{\theta}{\gamma}\right)\right)^{m}z_i e^{\gamma(1+h+l+r)(\lambda x)^{\beta}}$$

Applications

The GG-W distribution is applied to two real-world datasets to demonstrate its modeling capabilities. The performance is compared with other competing models using Statistical fit criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Log Likelihood (LL) and goodness-of-fit criteria like Kolmogorov-Smirnov (KS), Anderson-Darling (AD), and Cramér-von Mises (CM) statistics.

Consider an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba):

Table 1: Data of Breaking stress of carbon fibers (in Gba)

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

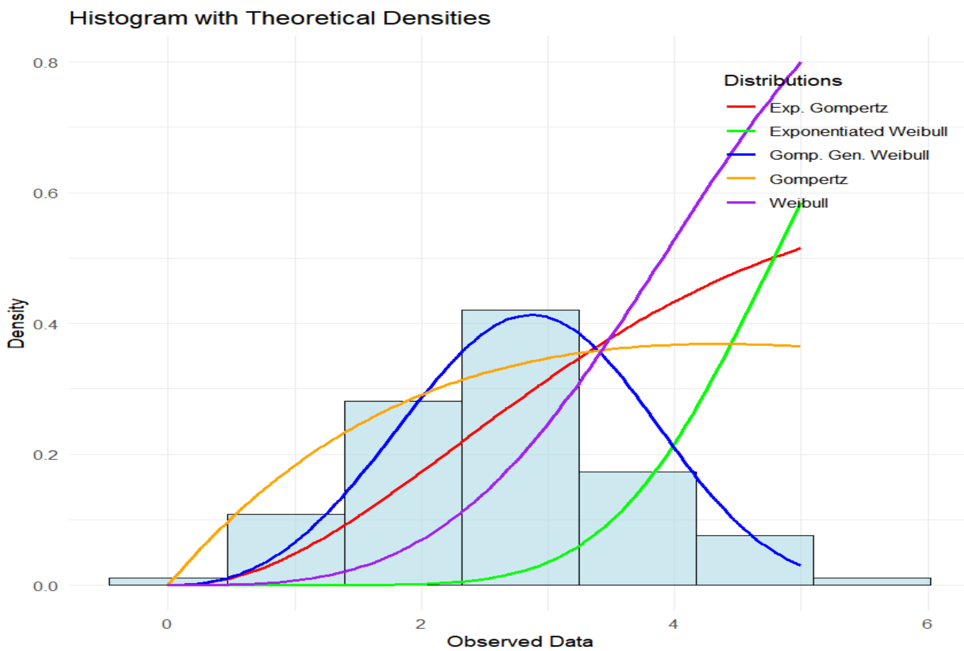
Table 2: Summary Statistics Breaking stress of carbon fibres (in GPa) Data

Mean	SD	Median	Min	Max	Skew	Kurtosis	Range
2.62	1.01	2.70	0.39	5.56	0.36	0.04	5.17

This table provides key descriptive statistics such as mean, standard deviation, median, minimum, maximum, skewness, kurtosis, and range**.**

Table 3: Log likelihood and Information Criterion of breaking stress of carbon fibres

Distribution	LogLikelihood	AIC	BIC
Exp. Gompertz	-295.6787	593.3574	595.9625
Exp. Weibull	-787.8326	1579.6652	1584.8755
Gompertz	-311.3294	626.6588	631.8692
Weibull	-944.2575	1894.5150	1902.3306
GGWeibull	-217.0381	444.0761	457.1020



DISCUSSION

The results presented in Table 3 show a clear comparison of the log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) values for several distributions fitted to the breaking stress data of carbon fibers. Among the models considered — including the Exponentiated Gompertz, Exponentiated Weibull, Gompertz, Weibull, and Gompertz Generalized Weibull (GGWeibull), the GGWeibull distribution clearly emerges as the best performer. It achieves the highest (least negative) log-likelihood value of −217.0381, compared to −295.6787 for the Exponentiated Gompertz, −311.3294 for the Gompertz, −787.8326 for the Exponentiated Weibull, and −944.2575 for the Weibull. In addition to superior log-likelihood, the GGWeibull also produces the lowest AIC and BIC values, with AIC = 444.0761 and BIC = 457.1020. These values are significantly lower than the next best model, the Exponentiated Gompertz (AIC = 593.3574, BIC = 595.9625), indicating that the GGWeibull achieves the best balance between model fit and complexity. The AIC and BIC are particularly important because they penalize the addition of unnecessary parameters, ensuring that a model's superior fit is not simply due to overfitting but rather reflects a genuine improvement in capturing the underlying data structure. The poor performance of the classical Weibull and Gompertz models, which show the worst log-likelihood, AIC, and BIC, underscores the limitations of traditional two-parameter distributions when applied to complex engineering data such as breaking stress. In contrast, the GGWeibull, derived by applying the Gompertz generator to the Weibull baseline, enhances flexibility and adaptability, making it well-suited to accommodate diverse shapes in hazard functions and tail behaviours that simpler models fail to capture. In practical terms, the superior fit of the GGWeibull distribution has meaningful implications for materials science and reliability engineering. Accurately modelling the stress distribution of carbon fibres is crucial for predicting material failure, optimising design, and improving safety margins. The GGWeibull's superior performance suggests it can provide more reliable estimates of failure probabilities, critical stress thresholds, and material durability compared to classical models. Overall, the analysis strongly supports the GGWeibull as the most suitable and effective model for this dataset, offering clear advantages over established alternatives in both statistical performance and practical application. The histogram with theoretical densities shown in the plot compares the fit of five distributions — Exponentiated Gompertz (red), Exponentiated Weibull (green), Gompertz Generalised Weibull (blue), Gompertz (orange), and Weibull (purple) — against the observed data of breaking stress of carbon fibres. Visually, the histogram (light blue bars) represents the empirical density of the observed data. Among the plotted curves, the blue curve (Gompertz Generalised Weibull) aligns most closely with the shape of the histogram, capturing both the central peak and the spread across the observed range. This supports the earlier statistical results where the GGWeibull model had the best log-likelihood, AIC, and BIC values, reflecting its superior fit. The red (Exponentiated Gompertz) and orange (Gompertz) curves rise more gradually and fail to capture the sharper central peak of the data, indicating underfitting in the main density region. The green (Exponentiated Weibull) curve rises sharply but overshoots in the upper range, suggesting poor alignment with the actual data distribution. The purple (Weibull) curve also sharply increases and peaks too early, showing a clear mismatch with the empirical histogram.

CONCLUSION

The analysis of the breaking stress data of carbon fibres clearly shows that the Gompertz Generalised Weibull (GGWeibull) distribution provides the best statistical and visual fit among all the considered models. This conclusion is supported by its highest log-likelihood and the lowest AIC and BIC values, indicating an optimal balance between model fit and complexity. The histogram with theoretical densities further confirms that the GGWeibull distribution closely matches the observed data pattern, accurately capturing both the central tendency and tail behaviour, while other distributions either underfit or overfit key areas of the data.

RECOMMENDATION

It is recommended that the Gompertz Generalised Weibull (GGWeibull) distribution be used for modeling breaking stress and related engineering datasets, as it offers superior flexibility and predictive performance compared to classical models. Practitioners and researchers should consider adopting the GGWeibull model for reliability analysis, failure prediction, and material strength studies. Additionally, future research should investigate the model's performance on larger and more diverse datasets and explore its applicability in other fields such as biomedical survival analysis, environmental risk assessment, and industrial process modeling.

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